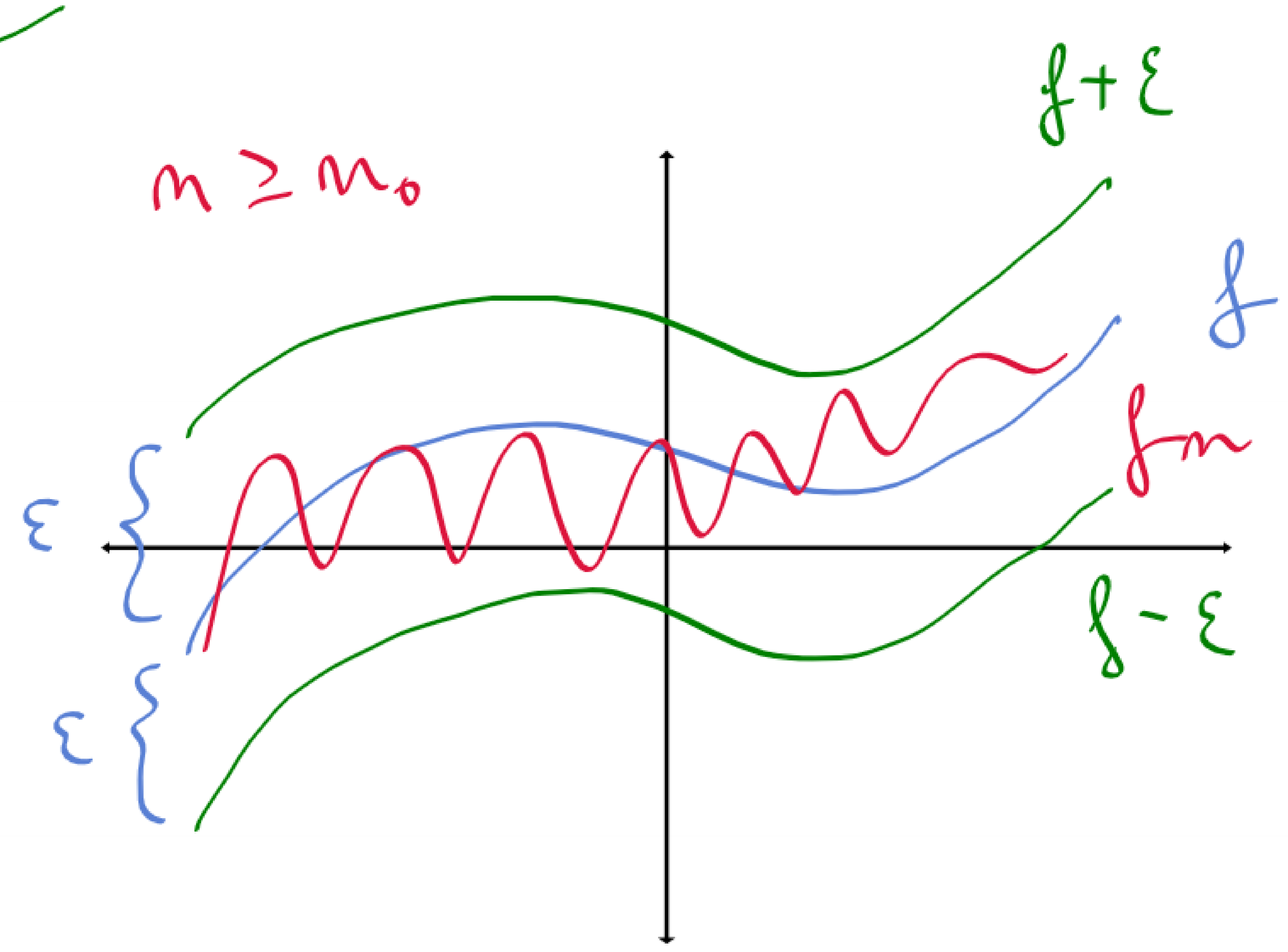
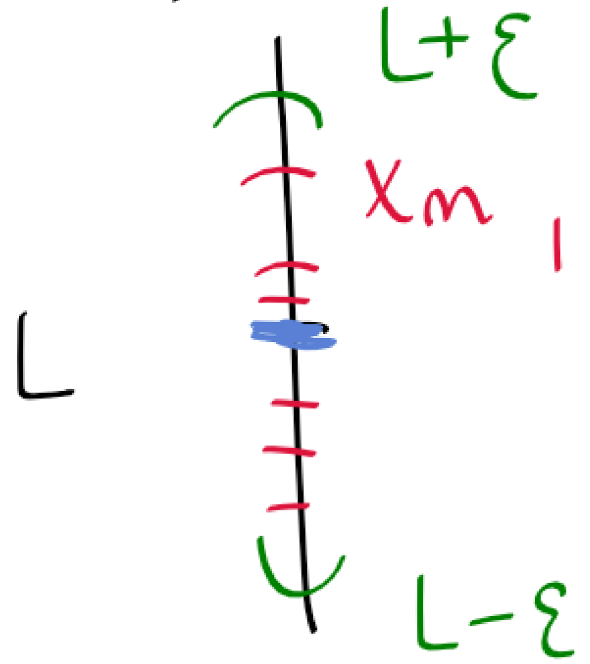
Stejnoměrná konvergence $f_n \rightrightarrows f$ na M $\overset{def.}{\iff}$

$\forall \varepsilon>0\ \exists n_0\in\mathbb{N}\ \forall n\geq n_0\ \forall x\in M:\ |f_n(x)-f(x)|<\varepsilon.$

BODOVÁ.



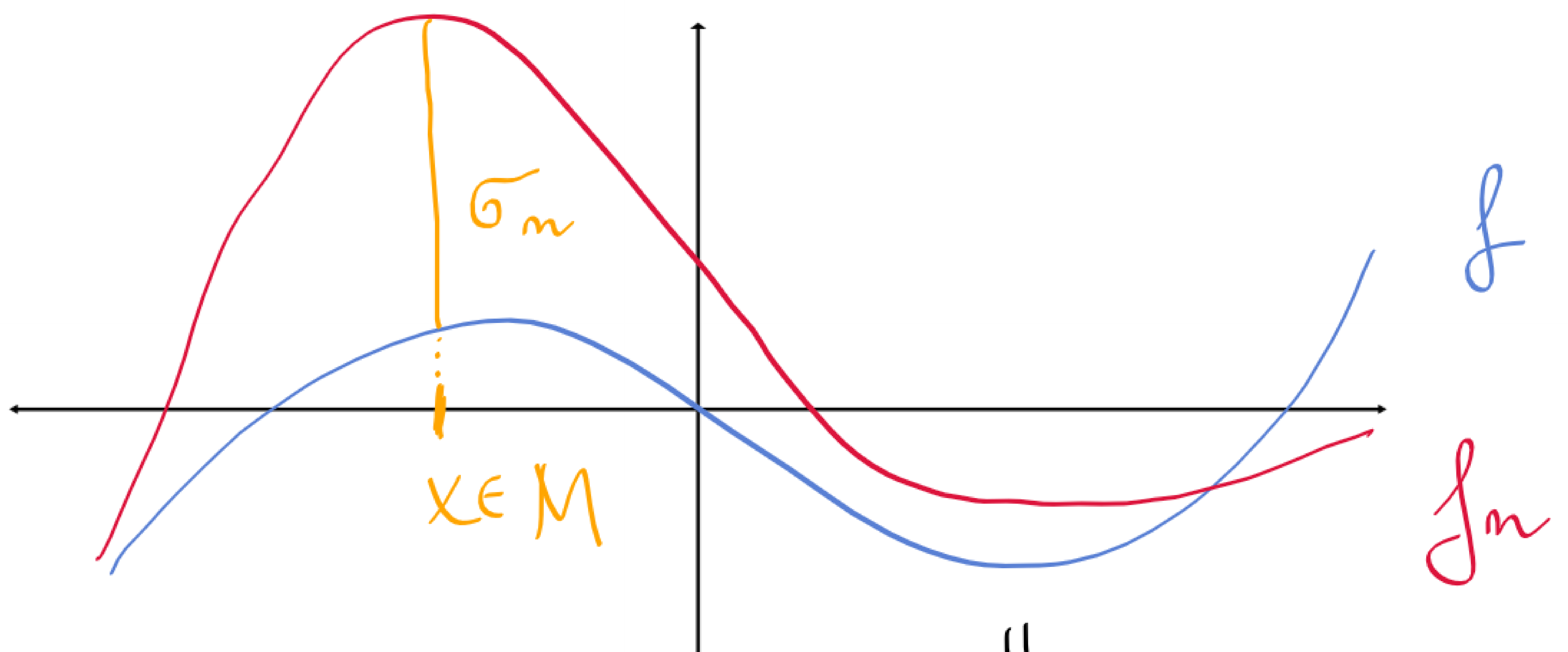
POZN.: lim. posl. $x_n \to L$



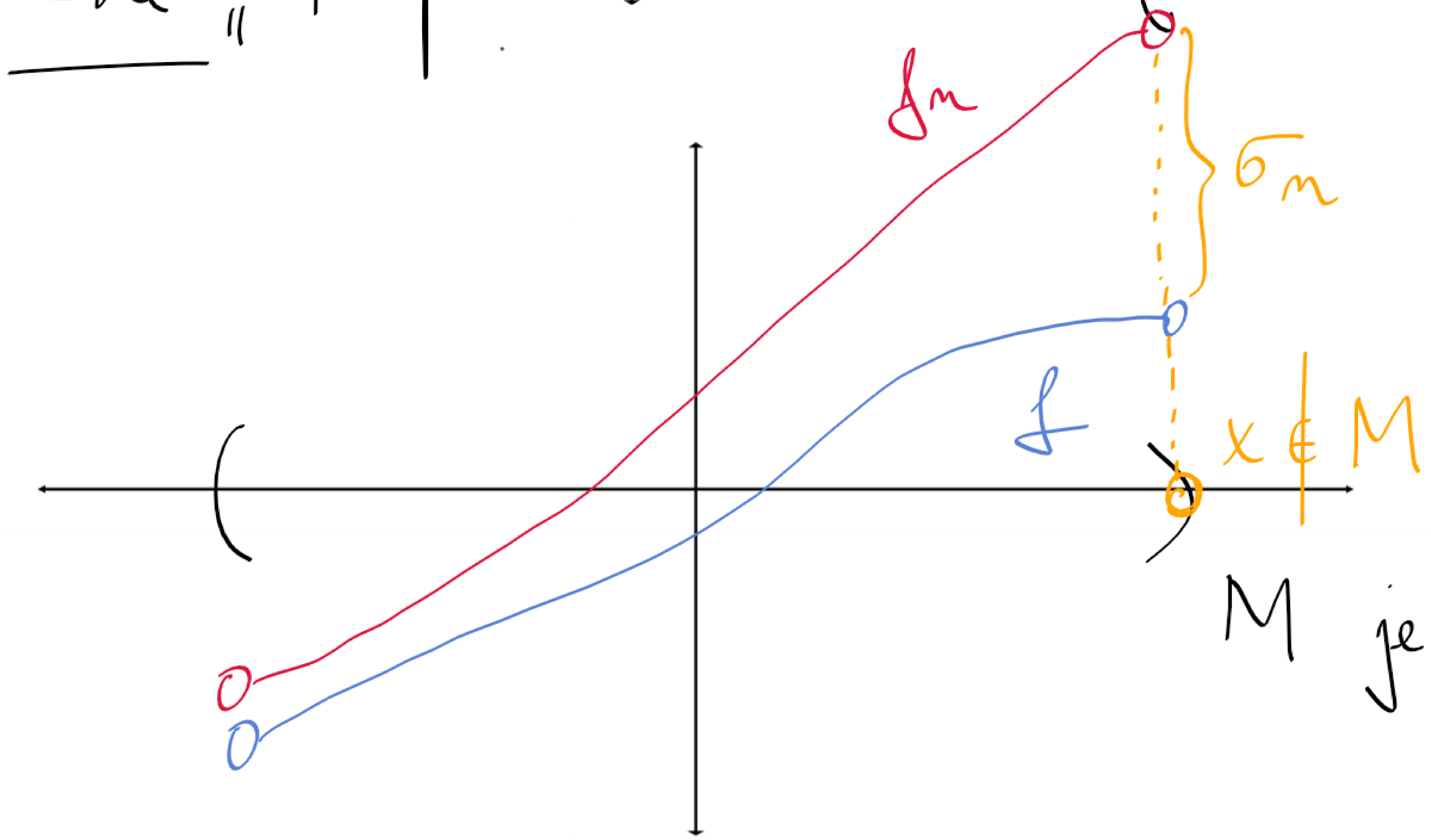
Lemma 73 (O σ_n). Nechť f, f_n $(n\in\mathbb{N})$ jsou definované aspoň na M.

Pak $f_n \rightrightarrows f$ na M $\iff$ $\lim\limits_{n\to\infty}\sigma_n = 0$, kde

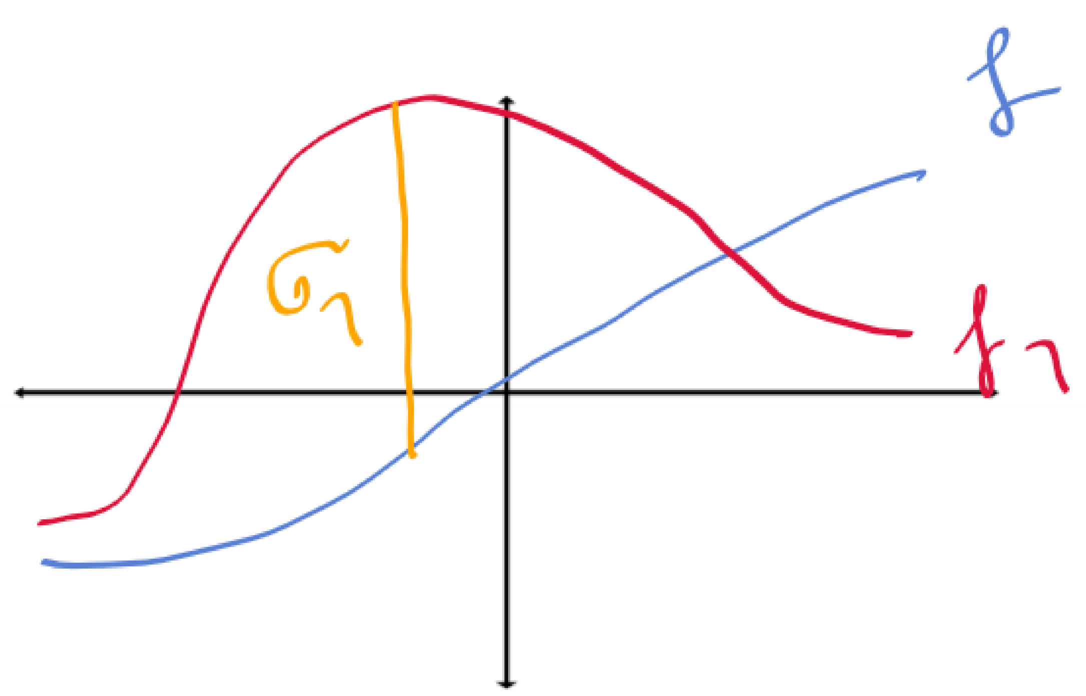
$$\sigma_n = \sup_{x\in M}|f_n(x)-f(x)|.$$

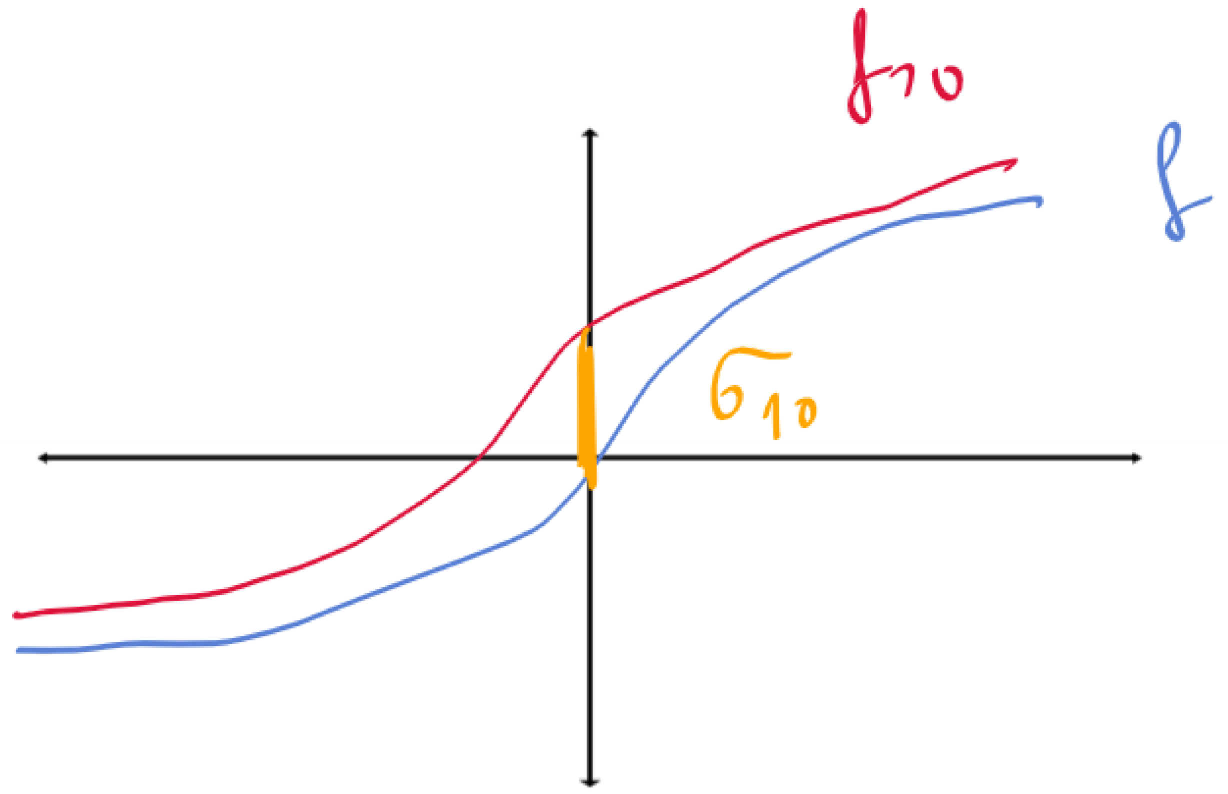


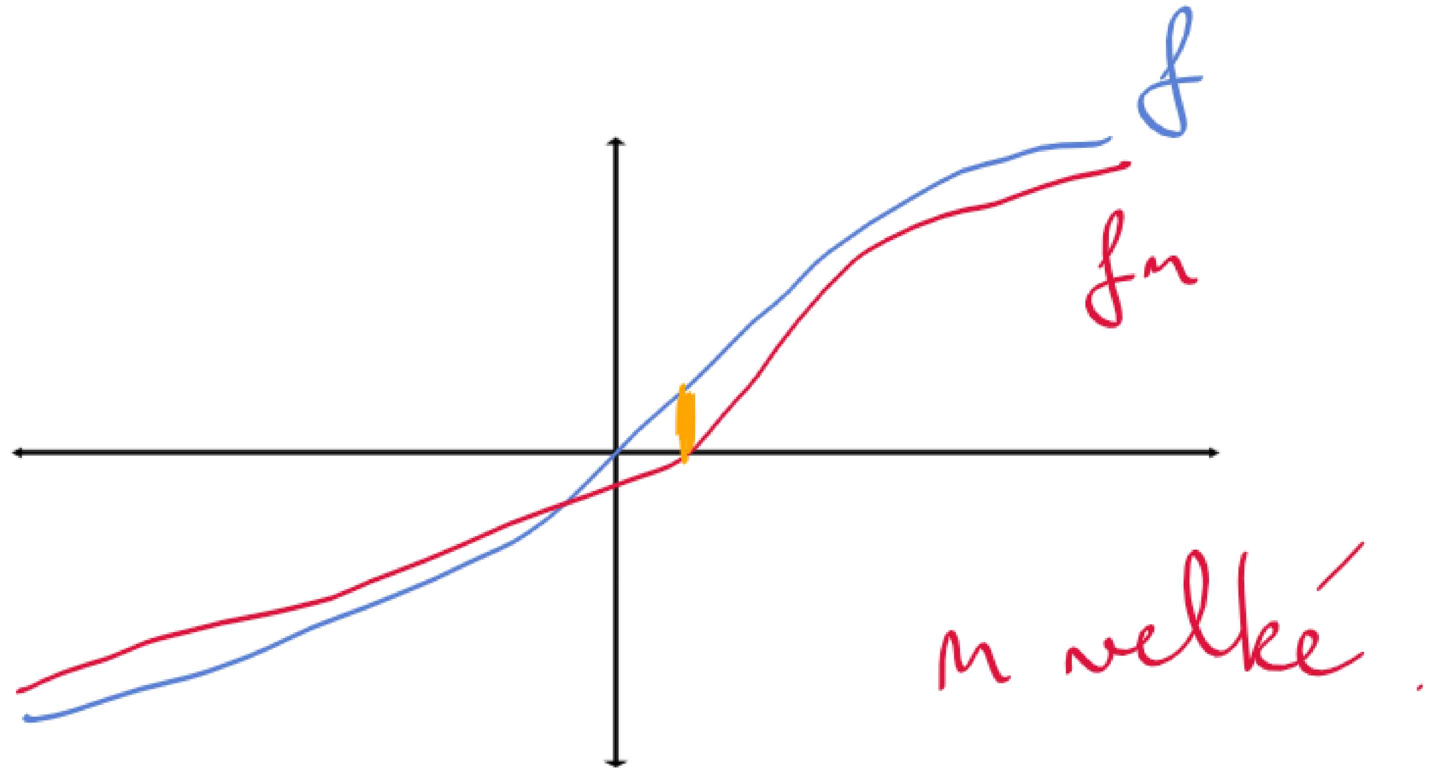
Zde „sup = max" ... největší vzdálenost.

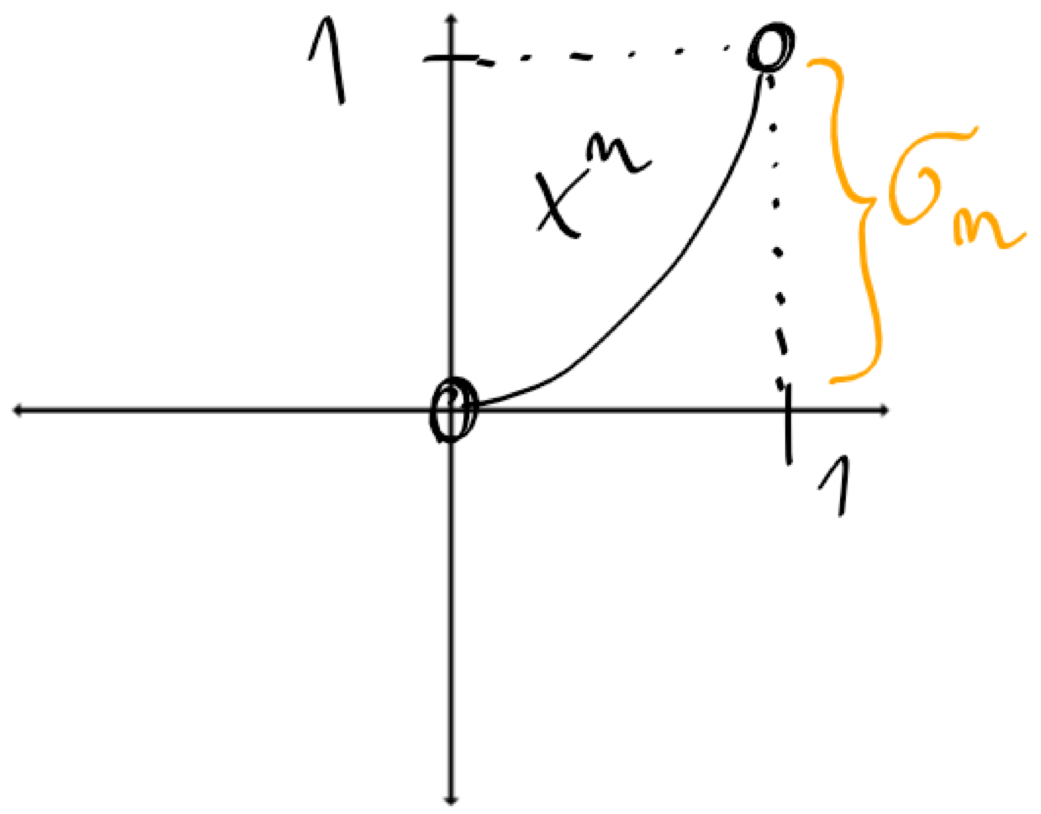


Zde max neexistuje, sup ano – – a má takovou hodnotu, jako by $x\in M$. (a šlo o max).









Příklad: $f_n(x) = x^n$, $x \in (0,1)$.

1. KROK: Určit funkci f (limitní fci) jakožto bodovou limitu f_n.

x bude pevné z $(0,1)$: Pro toto x počítejme

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0 =: f(x)$$

To jest bodová limita je $f \equiv 0$ na $(0,1)$.

2. KROK Jestlipak $f_n \rightrightarrows f$? (na $(0,1)$)

Spočítáme pro libovolné pevné n: $\sigma_n = ?$

$$\sigma_n \overset{\text{def.}}{=} \sup_{x\in(0,1)} |f_n(x) - f(x)| =$$

$$= \sup_{x\in(0,1)} |x^n - 0| = \sup_{x\in(0,1)} x^n =$$

x^n rostoucí

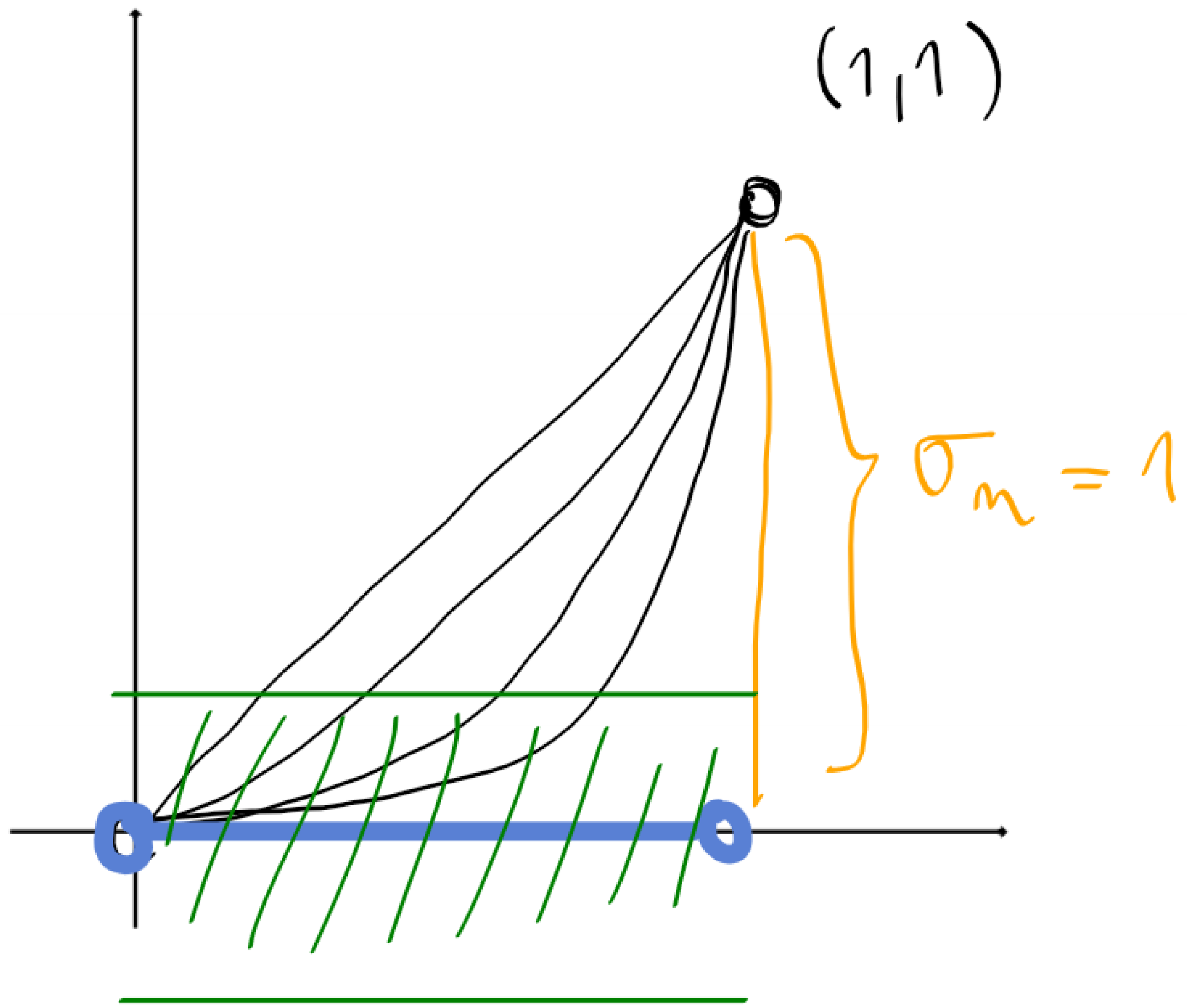
$$= 1^n = 1.$$

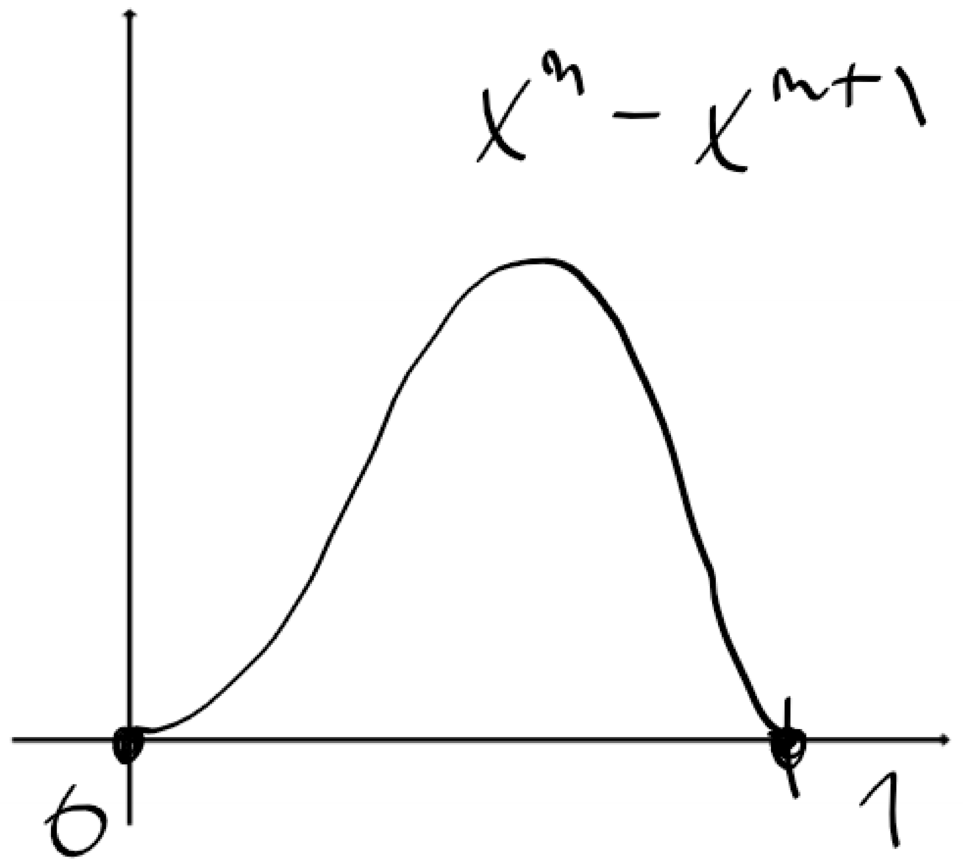
Tedy $\forall n \in \mathbb{N}$: $\sigma_n = 1$.

3. KROK: Spočítáme $\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} 1 = 1$.

Závěr tedy je (L73): $f_n \not\rightrightarrows f = 0$.

Protože f je jediný kandidát, $f_n \not\rightrightarrows$.





$\boxed{x \in (0,1)}$

Příklad: $f_n(x) = x^n - x^{n+1}$

1. KROK: $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \big(\overset{\nearrow 0}{x^n} - \overset{\nearrow 0}{x^{n+1}}\big) = 0 - 0 = 0$

pro $x \in (0,1)$. Tj. $f(x) \equiv 0$ na $(0,1)$.

2. KROK: $\sigma_n = \sup\limits_{x\in(0,1)} |f_n(x) - f(x)| =$

$$= \sup_{x\in(0,1)} \big|\underbrace{x^n - x^{n+1}}_{>0 \Leftarrow [x^{n+1} < x^n]}\big| = \sup_{x\in(0,1)} (x^n - x^{n+1}).$$

Chceme tedy vlastně maximum funkce $x^n - x^{n+1}$ na $[0,1]$. (Tj. $\sup\limits_{x\in(0,1)}$.)

$x = 0$ ... 0 $\quad$ $x = 1$ ... $1^n - 1^{n+1} = 0$

$(x^n - x^{n+1})' = n x^{n-1} - (n+1)x^n = 0$

$\iff x^{n-1}(n - (n+1)x) = 0$

$\iff x = 0 \ \vee\ (n+1)x = n$

$\iff \boxed{x = 0} \ \vee\ x = \frac{n}{n+1}.$

nezájem – hodnota je 0.

Z obrázku je jasné, že $x = \frac{n}{n+1}$ je bodem maxima fce $x^n - x^{n+1}$.

$\sigma_n = \left(\frac{n}{n+1}\right)^n - \left(\frac{n}{n+1}\right)^{n+1} = \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right)$

$$= \left(\frac{n}{n+1}\right)^n \left(1-\frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} = \sigma_n$$

3. KROK: $\lim\limits_{n\to\infty} \sigma_n =$

$$= \lim_{n\to\infty} \underbrace{\left(\frac{n}{n+1}\right)^n}_{\text{omezená}} \cdot \underbrace{\frac{1}{n+1}}_{\to 0} = 0.$$

$$0 < \frac{n}{n+1} < 1$$

$$0 < \left(\frac{n}{n+1}\right)^n < 1 \implies \text{OMEZENOST.}$$

Závěr: $\lim\limits_{n\to\infty} \sigma_n = 0 \overset{L73}{\implies} f_n \rightrightarrows f$ na $(0,1)$.

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim \frac{1}{\left(\frac{n+1}{n}\right)^n} =$$

$$= \frac{1}{\lim\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$$

Resp.: $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n \overset{H.V.}{=} \lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x =$

$$= \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)} = e^1 = e.$$

$$\lim x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \overset{\text{VOLSF}}{=} 1.$$

$\frac{1}{x} \to 0$

známá pro ln

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Příklad: $f_n(x) = \dfrac{nx}{1+n+x}$, $x \in [0,1]$

1. KROK: (x pevné) $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{nx}{1+n+x} =$

$$= \lim \frac{\cancel{n}\cdot x}{\cancel{n}\left(\frac{1}{n}+1+\frac{x}{n}\right)} = \frac{x}{0+1+0} = x =: f(x)$$

2. KROK: (n pevné) $\sigma_n = \sup\limits_{x\in[0,1]} |f_n(x) - f(x)| =$

$$= \sup_{x\in[0,1]} \left| \frac{nx}{1+n+x} - x \right| =$$

$$= \sup \left| x \cdot \left(\frac{n}{1+n+x} - 1 \right) \right| =$$

$$= \sup \left| x \cdot \frac{-1-x}{1+n+x} \right| =$$

$$= \sup \left| (-1) \cdot \frac{x\cdot(1+x)}{\underbrace{1+n+x}_{\geq 0,\ x\in[0,1]}} \right| =$$

$$= \sup \frac{x(1+x)}{1+n+x} = \max_{x\in[0,1]} \underbrace{\frac{x(1+x)}{1+n+x}}_{g_n(x)}$$

$$g_n'(x) = \left(\frac{x^2+x}{1+n+x} \right)' =$$

$$= \frac{(2x+1)(1+n+x) - (x^2+x)(1)}{(1+n+x)^2} =$$

$$= \frac{1}{(1+n+x)^2}\left(\cancel{2x} + \cancel{2n}x + 2x^2 + 1 + n + \cancel{x} - x^2 \cancel{- x} \right)$$

$$= \frac{1}{(1+n+x)^2}\left(x(2+2n) + x^2 + 1 + n \right) = 0 \iff$$

$$x(2+2n) + x^2 + 1 + n = 0$$

$$x^2 + (2+2n)x + (1+n) = 0$$

$$x_{1,2} = \frac{-(2+2n) \pm \sqrt{(2+2n)^2 - 4(1+n)}}{2} =$$

$$= \frac{-(2+2n) \pm \sqrt{4\cdot(1+n)^2 - 4(1+n)}}{2} =$$

$$= -1-n \pm \sqrt{(1+n)^2 - (1+n)} =$$

nezájem, $\notin [0,1]$

$$= -1-n \pm \sqrt{(1+n)\cdot n}$$

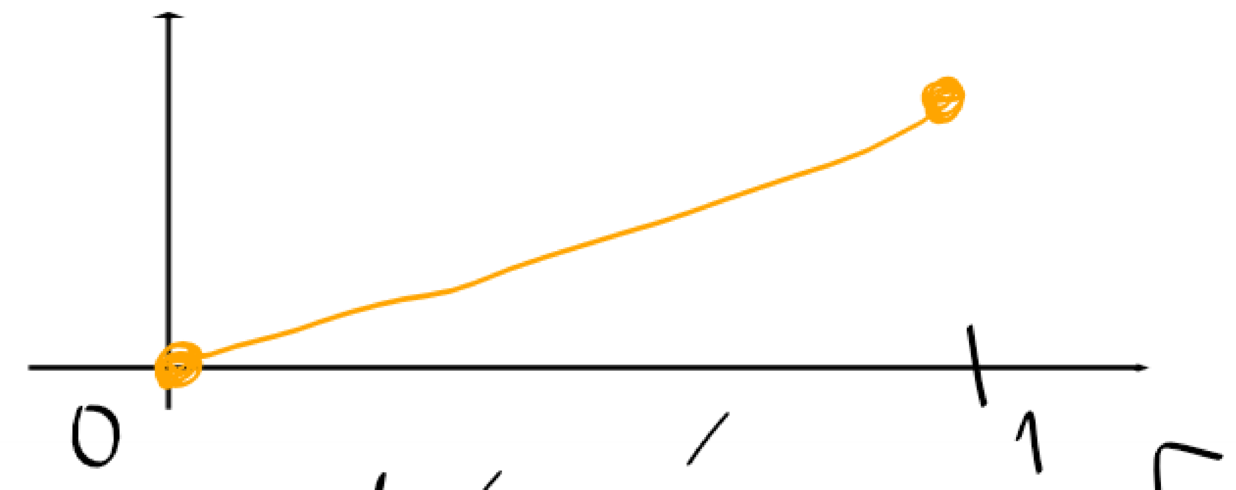
Tj. jediný podezřelý bod je

$$x_n = \sqrt{n\cdot(1+n)} - 1 - n <$$

$$< \sqrt{(n+1)(n+1)} - 1 - n = n+1 - 1-n = 0$$

Tedy oba kořeny jsou < 0.

Tedy $g_n(x)$:



- $g_n(0) = 0$
- nezáporná a monotónní na $[0,1]$.
- $g_n(1) > 0$.

Celkem: g_n je neklesající (dokonce rostoucí) na $[0,1]$.

Tedy $$\sup_{x\in[0,1]} g_n(x) = \left(\max g_n(x)\right) =$$

$$= g_n(1) = \frac{1\cdot(1+1)}{1+n+1} = \frac{2}{n+2} = \sigma_n$$

3. KROK: $\lim \sigma_n = \lim \frac{2}{n+2} = 0$

$\overset{L73}{\Longrightarrow} f_n \rightrightarrows f$ na $[0,1)$ (kde $f(x)=x$

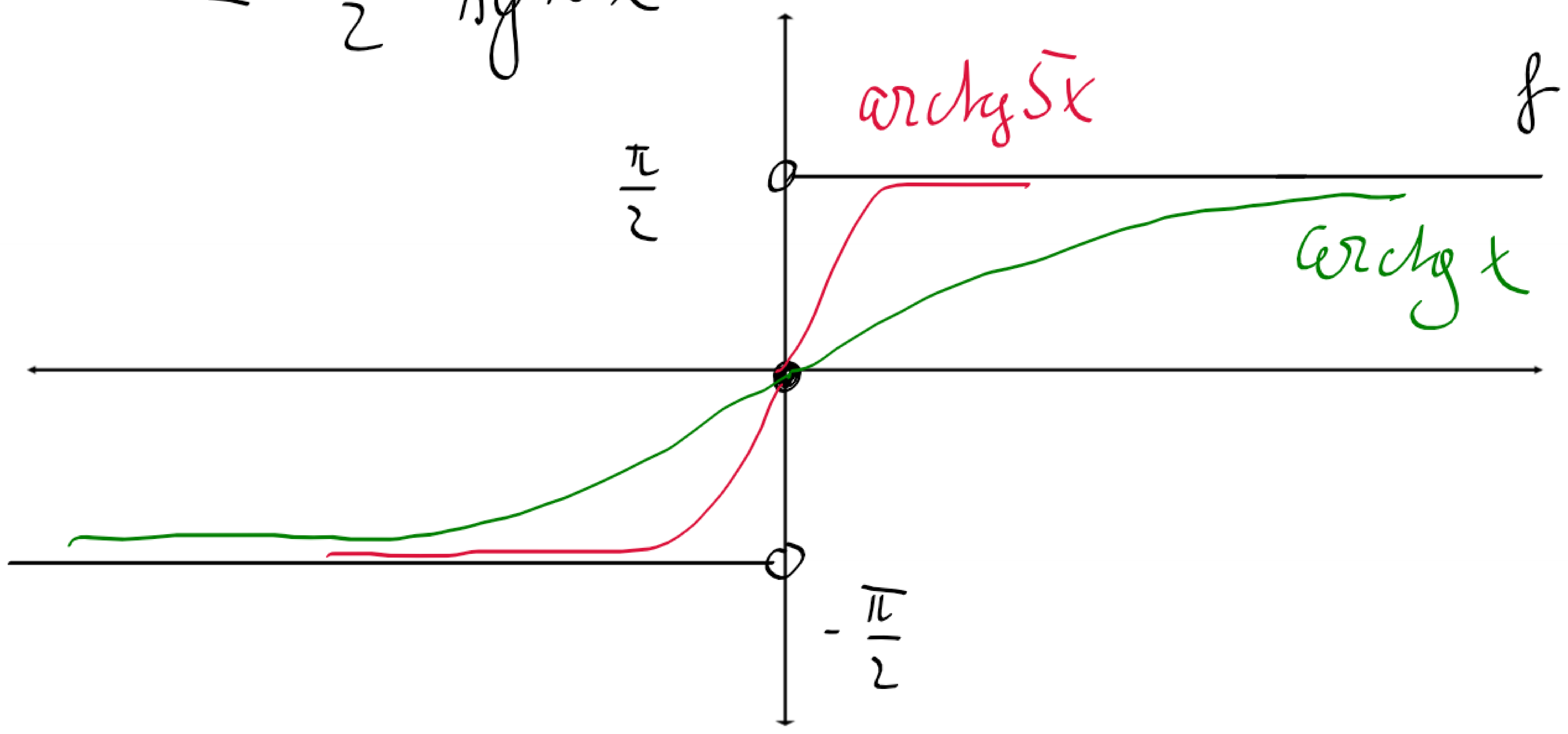
Příklad: $f_n(x) = \operatorname{arctg}(nx)$, $x \in \mathbb{R}$

1. KROK: Bodová limita, x je pevné

$$f(x) = \lim_{n\to\infty} f_n(x) \dots$$

$$= \lim_{n\to\infty} \operatorname{arctg}(nx) = \begin{cases} \frac{\pi}{2}, & x>0 \\ 0, & x=0 \\ -\frac{\pi}{2}, & x<0 \end{cases}$$

$$= \frac{\pi}{2} \cdot \operatorname{sgn} x$$



Cv.: $\sigma_n = \dots \frac{\pi}{2} \not\to 0$

Závěr: $f_n \not\rightrightarrows f$ na $\mathbb{R}$.

Všimněme si, že $\forall n \in \mathbb{N}$: f_n je spoj.

Ale $f_n \to f$ a f není spoj.

Ale „$\rightrightarrows$" zachovává spojitost.

Tedy zde nemůže nastávat „$\rightrightarrows$", neboť spojitost nebyla zachována.

(totiž limita spoj. fcí je zde nespoj.!)